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Second order expansion of action functionals of noncommutative gauge theories

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Abstract: Field theory and gauge theory on noncommutative spaces have been established as their own areas of research in recent years. The hope prevails that a noncommutative gauge theory will deliver testable experimental predictions and will thus be a serious candidate for an extension of the Standard Model. This note contains the results for expanded gauge theory actions on a noncommutative space with constant $\theta^{\mu\nu}$, up to second order, together with a discussion of the ambiguities of the expanded theory and how they affect the action.

Keywords: Brane Dynamics in Gauge Theories, Non-Commutative Geometry, Beyond Standard Model.

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1. Introduction

Noncommutative (NC) gauge theory has been an intensively discussed issue during the past five years, prompted by the results of Seiberg and Witten's seminal paper [1], connecting commutative and NC gauge theory (on the canonical NC space, see below). Most of the work on this subject covered the all orders or summed-up NC gauge theory, typically discussed in the string theory context. Recent results [2, 3] show that NC spacetime might be endowed with a deformed symmetry structure. Still, the phenomenon of so-called ultraviolet-infrared mixing [4]–[6] causes concerns to treat the summed-up NC field theory as a serious candidate for a realistic extension of the Standard Model of elementary particle physics.

It has been shown in a series of papers [7]–[9] that the results of [1] can also be obtained in a setting entirely independent of string theory. The ansatz here uses only algebraic properties of the canonical NC space via the properties of the $\star$-product. The canonical NC space is characterised by noncommuting coordinate functions $[\hat{x}^\mu, \hat{x}^\nu] = \theta^{\mu\nu}$ with a constant background field $\theta^{\mu\nu}$. This NC space can be realised in terms of ordinary functions by means of the $\star$-product $(f \star g)(x) = \exp(\frac{i\hbar}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu})f(y)g(z)|_{y,z\to x}$.

The most important result of [9] is that such a purely constructive ansatz works. The $\star$-product is treated in this ansatz in an expanded way (partially up to second order in [9]). Because of the expanded approach the conceptual problems related to the UV/IR mixing are avoided, since an effective IR-regulator is introduced. The expanded approach introduces and uses enveloping algebra-valued gauge theory. Therefore no restrictions exist on the admissible gauge groups, especially $SU(N)$-gauge theory can be realised on the NC space. Thus, a NC generalisation of the Standard Model has been constructed [10]. While no statements about uniqueness or existence can be made a priori, the approach can be formalised and existence can be proven [11].

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Several authors have analysed such an expanded version of a NC Standard Model with a view to derive phenomenological consequences. The important observation of [10] has been that the expanded NC Standard Model does not suffer from the physically problematic charge quantisation that had been observed in the non-expanded NC gauge theory [12, 13]. Also the tensor product of several gauge groups, necessary for a generalisation of the Standard Model, can be treated in a satisfactory way in this context. Again, this issue had been highly non-trivial in the non-expanded approach [14].

The key phenomenological result is that the expanded approach leads to new non-linear couplings between the gauge bosons of the different gauge sectors of the Standard Model. Also matter fields acquire new multiple couplings to gauge bosons that are classically prohibited, e.g. neutral particles may couple to the photon because of the NC geometry. Therefore this model predicts higher-dimensional operators which are clearly power-counting non-renormalisable. This is acceptable, if the theory is regarded as an effective physical theory, presupposing other physics at some higher scale.

Some phenomenological studies, e.g. [15], treat $\theta^{\mu\nu}$ as an ether-like Lorentz invariance violating field pervading spacetime. The calculated experimental bounds are very high. In contrast, $\theta^{\mu\nu}$-expanded models should be regarded as IR-regularisable. Two sorts of phenomenological consequences can be distinguished: in [16] and [17] the pure gauge sector has been investigated, Standard Model-forbidden vertices of three (possibly different) gauge bosons have been found. This means that the Z boson could decay into two γ or two gluons, which might well be measurable. However, due to an ambiguity in the choice of the matrix trace the pure gauge sector of NC gauge theory is not very well suited to derive strict bounds on the NC scale. Ongoing analysis manages to constrain the trace in the electroweak sector [17], still this freedom limits predictiveness. Therefore, much energy has been devoted as well to studies of $\theta^{\mu\nu}$-expanded NC effects in the fermionic sector [18]–[21]. Yet, as has been argued already [22], great care has to be taken to evaluate the bounds on the NC scale by such analyses.

In any case, all phenomenological analyses center on effects deriving from first-order terms in $\theta^{\mu\nu}$. The concise calculation of second order terms in $\theta^{\mu\nu}$ has so far been ignored due to the technical complexity of some calculations. This work fills this gap. We recall the outline of the basic approach first developed in [9] and proceed to presenting full results up to second order in $\theta^{\mu\nu}$ in the action functionals.

In the meantime, many discussions have continued the work started in [9], e.g. the quantisation of this expanded model has been discussed [23, 24]. We choose among the possible actions for the NC gauge theory those which are direct generalisations of the actions of the Standard Model of particle physics. We will omit the discussion of other interesting actions such as Born-Infeld (but the approach would be analogous).

With this restriction to Standard Model-type actions we could of course miss out on models which might be more suited to NC spaces, cp. [25]. Also we omit pure scalar field theory, the Einstein-Hilbert action and supersymmetric actions. These models have been discussed by now in the literature, e.g. [26, 27]. We have also applied this approach to other NC spaces, such as the κ-Minkowski spacetime [28].

2. Enveloping algebra-valued gauge theories

We start by fixing the notations for infinitesimal gauge transformations on a commutative space [29]:

$$\delta_\alpha \psi_i^0(x) = i\alpha_a(x)(T^a)_{ij}\psi_j^0(x)\,. \tag{2.1}$$

The field $\psi^0(x)$ is in the fundamental representation of an arbitrary non-abelian gauge group. Abelian simplifications are not spelt out. We can considerably simplify the notation of (2.1), keeping the generators of the gauge Lie algebra and the x-dependence of ψ^0 and α implicit, $\alpha \equiv \alpha_a(x)T^a$ and $\psi^0 \equiv \psi^0(x)$:

$$\delta_\alpha \psi^0 = i\alpha\psi^0\,. \tag{2.2}$$

Since the generators T^a form a Lie algebra $[T^a, T^b] = if_c^{ab}T^c$, the commutator of two infinitesimal gauge transformations closes:

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi^0 = \alpha\beta\psi^0 - \beta\alpha\psi^0 = \delta_{-i[\alpha,\beta]}\psi^0\,. \tag{2.3}$$

A gauge transformation acts on the conjugate transpose of a field $\delta_\alpha\overline{\psi^0} = -i\overline{\psi^0}\alpha$, such that $\delta_\alpha(\overline{\psi^0}\psi^0) = 0$. Derivatives $\partial_\mu\psi^0$ can be rendered gauge covariant by adding a gauge potential:

$$\delta_\alpha(\mathcal{D}_\mu^0\psi^0) = \delta_\alpha((\partial_\mu - iA_\mu^0)\psi^0) \stackrel{!}{=} i\alpha(\mathcal{D}_\mu^0\psi^0)\,, \quad \Rightarrow \quad \delta_\alpha A_\mu^0 = \partial_\mu\alpha - i[A_\mu^0, \alpha]\,. \tag{2.4}$$

The gauge potential transforms in the adjoint representation, as does the field strength $F_{\mu\nu}^0$, which is constructed from the commutator of two covariant derivatives

$$\begin{aligned} F_{\mu\nu}^0 = i[\mathcal{D}_\mu^0, \mathcal{D}_\nu^0] = \partial_\mu A_\nu^0 - \partial_\nu A_\mu^0 - i[A_\mu^0, A_\nu^0]\,, \quad &\Rightarrow \quad \delta_\alpha F_{\mu\nu}^0 = i[\alpha, F_{\mu\nu}^0], \\ [\mathcal{D}_\lambda^0, F_{\mu\nu}^0] + [\mathcal{D}_\mu^0, F_{\nu\lambda}^0] + [\mathcal{D}_\nu^0, F_{\lambda\mu}^0] &= 0 \quad \text{(Bianchi identity)}\,. \end{aligned} \tag{2.5}$$

In analogy, we start the analysis of gauge theory on NC space based on infinitesimal gauge transformations. The NC space is represented on the algebra of functions of commutative variables by a $\star$-product. Therefore (2.1) is replaced by[1]

$$\delta_\Lambda\psi(x) = i\Lambda(x) \star \psi(x)\,. \tag{2.6}$$

As before, the gauge transformation of the field $\psi(x)$ is implemented by left $\star$-multiplication of $\psi(x)$ with a function $\Lambda(x)$. This is again a local gauge transformation, although this statement has to be taken with a grain of salt, since the $\star$-product involves an arbitrary number of derivatives and is therefore highly non-local. We will use the $\star$-product always in an expanded form, thus it is local at every order of the expansion (this does not imply all-orders locality). Representations for which a field $\psi(x)$ is multiplied from the right are possible as well. The other quantities of NC gauge theory are defined in the following way: Since the Moyal-Weyl $\star$-product is x-independent, it commutes with derivatives. Therefore we define on the canonically NC space as generalisations of (2.4) and (2.5) a NC covariant

[1]For convenience we choose the less correct notation $\Lambda(x) \star \psi(x)$ instead of $(\Lambda \star \psi)(x)$.

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derivative along with a NC gauge potential and the NC field strength:

$$\begin{aligned}
\delta_\Lambda(\mathcal{D}_\mu\psi) = \delta_\Lambda(\partial_\mu\psi - iA_\mu \star \psi) \stackrel{!}{=} i\Lambda \star \mathcal{D}_\mu\psi\,, \quad &\Rightarrow \quad \delta_\Lambda A_\mu = \partial_\mu\Lambda - i[A_\mu \stackrel{\star}{,} \Lambda] \\
F_{\mu\nu} = i[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu \stackrel{\star}{,} A_\nu]\,, \quad &\Rightarrow \quad \delta_\Lambda F_{\mu\nu} = i[\Lambda \stackrel{\star}{,} F_{\mu\nu}]\,, \\
[\mathcal{D}_\lambda \stackrel{\star}{,} F_{\mu\nu}] + [\mathcal{D}_\mu \stackrel{\star}{,} F_{\nu\lambda}] &+ [\mathcal{D}_\nu \stackrel{\star}{,} F_{\lambda\mu}] = 0 \;\; \text{(Bianchi identity)}\,.
\end{aligned} \tag{2.7}$$

For a Lie algebra-valued gauge parameter $\Lambda(x) = \Lambda_a(x)T^a$, two gauge transformations in general do not close anymore as in (2.3)

$$\begin{aligned}
(\delta_{\Lambda_1}\delta_{\Lambda_2} - \delta_{\Lambda_2}\delta_{\Lambda_1})\psi(x) \;=\; &\frac{1}{2}[T^a, T^b]\{\Lambda_{1,a}(x) \stackrel{\star}{,} \Lambda_{2,b}(x)\} \star \psi(x) + \\
&+ \frac{1}{2}\{T^a, T^b\}[\Lambda_{1,a}(x) \stackrel{\star}{,} \Lambda_{2,b}(x)] \star \psi(x)\,.
\end{aligned} \tag{2.8}$$

Only a $\mathrm{U}(N)$ gauge theory allows to express the anti-commutator $\{T^a, T^b\}$ again in terms of the generators T^a [1]. The only alternative is that the concept of Lie algebra gauge theories has to be generalised, towards the enveloping algebra of the Lie algebra we started with. The Lie algebra and its enveloping algebra are the only mathematical entities, which are consistent with $[T^a, T^b] = if^{ab}_c T^c$, independent of a specific representation. The enveloping algebra $\mathcal{A}_T$ of the Lie algebra is an infinite-dimensional algebra freely generated by T and divided by the ideal generated by $[T^a, T^b] = if^{ab}_c T^c$ [8]. It consists of all (symmetrically) ordered tensor powers of the generators T^a and therefore is an infinite-dimensional tensor algebra.

Since the anti-commutator of (2.8) is in the enveloping algebra as well, the commutator of two enveloping algebra-valued gauge transformations remains enveloping algebra-valued. However, $\mathcal{A}_T$ has an infinite number of components, expanding order by order. This means that an enveloping algebra-valued gauge theory would have an infinite number of degrees of freedom. This infinite number can be reduced demanding that the expansion coefficients of enveloping algebra-valued quantities at every order depend only on the Lie algebra-valued quantities, i.e. the zeroth order. Then the NC gauge theory is determined entirely by the gauge theory of the commutative space, with the same number of degrees of freedom.

We construct the enveloping algebra gauge theory in an expanded way. To perform the construction we use as a starting point the consistency condition that two consecutive gauge transformations have to close into another one (2.3). We find that the commutative gauge potential A^0_μ appears in the expansion of all quantities of NC gauge theory:

$$\Lambda_\alpha := \Lambda[\alpha, A^0_\mu], \qquad \psi := \psi[\psi^0, A^0_\mu] \qquad A_\mu := A_\mu[A^0_\mu], \quad \text{and} \quad F_{\mu\nu} := F_{\mu\nu}[A^0_\mu]\,, \tag{2.9}$$

where the square brackets denote functional dependence. In particular, these quantities depend on the Lie algebra quantities and an arbitrary number of derivatives on them. Still, the functionals are supposed to be local in the sense that at any finite order in the expansion, only a finite number of derivatives appears. To avoid notational clutter, we will keep this functional dependence implicit.

In this constructive approach, there is no principle to ensure that the reduction to the commutative degrees of freedom is indeed possible. We have to perform an explicit construction order by order in the parameter of the noncommutativity.

Since Λ_α depends on Lie algebra quantities only, δ_{Λ_α} reduces to $\delta_{\Lambda_\alpha} = \delta_\alpha$. Since Λ_α depends on A^0_μ explicitly, a gauge transformation of Λ_α is not zero $\delta_\alpha \Lambda_\beta \neq 0$. Therefore the consistency relation (2.3) for enveloping algebra-valued gauge theory, which must be true for all fields ψ, reads [9]:

$$i\delta_\alpha \Lambda_\beta - i\delta_\beta \Lambda_\alpha + [\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta] = i\Lambda_{-i[\alpha,\beta]} \,. \tag{2.10}$$

This consistency condition has the virtue of being an equation of the gauge parameter Λ_α alone (in contrast to [1]). Once we find solutions for (2.10), it is possible to solve

$$\delta_\alpha \psi = i\Lambda_\alpha \star \psi \,, \qquad \text{and} \qquad \delta_\alpha A_\mu = \partial_\mu \Lambda_\alpha - i[A^\mu \stackrel{\star}{,} \Lambda_\alpha] \,, \quad \text{etc.} \tag{2.11}$$

Since the enveloping algebra was introduced because of the NC $\star$-product, we expand Λ_α in terms of the parameter of the noncommutativity $\hbar$:

$$\Lambda_\alpha = \alpha + \hbar \Lambda^1_\alpha + \hbar^2 \Lambda^2_\alpha + \cdots \,. \tag{2.12}$$

Correspondingly, ψ, A_μ and $F_{\mu\nu}$ are expanded in terms of $\hbar$

$$\psi = \psi^0 + \hbar\psi^1 + \hbar^2 \psi^2 + \cdots \,, \tag{2.13}$$

$$A_\mu = A^0_\mu + \hbar A^1_\mu + \hbar^2 A^2_\mu + \cdots \,, \qquad \text{etc.} \tag{2.14}$$

In particular, we do not expand in the basis of the enveloping algebra. There are other suitable expansions, e.g. in terms of the number of factors of gauge potentials A^0_μ. This expansion allows some interesting all-orders summations [9, 30].

3. $\theta^{\mu\nu}$-expanded fields up to second order

Starting from (2.10), we expand the $\star$-product

$$f(x) \star g(x) = \exp\left(\frac{i\hbar}{2}\theta^{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial z^\nu}\right) f(y)g(z)|_{y,z\to x} \tag{3.1}$$

order by order in $\hbar$, solve the resulting expanded version of (2.10) at a given order, reinsert the solution for calculating the next order etc. The expansions (2.12), (2.13) and (2.14) are defined in such a way that (2.11) coincides in zeroth order with (2.2) and (2.4).

We expand (2.10) to first order in $\hbar$

$$i(\delta_\alpha \Lambda^1_\beta - \delta_\beta \Lambda^1_\alpha) + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] + [\alpha \stackrel{\star}{,} \beta]|_{\mathcal{O}(\hbar^1)} = i\Lambda^1_{-i[\alpha,\beta]}, \tag{3.2}$$

and insert the explicit form of the Moyal-Weyl $\star$-product to obtain

$$\Delta\Lambda^1 = i\left(\delta_\alpha \Lambda^1_\beta - \delta_\beta \Lambda^1_\alpha\right) + [\alpha, \Lambda^1_\beta] + [\Lambda^1_\alpha, \beta] - i\Lambda^1_{-i[\alpha,\beta]} = -\frac{i\hbar}{2}\theta^{\mu\nu}\{\partial_\mu\alpha, \partial_\nu\beta\} \,. \tag{3.3}$$

This is an inhomogeneous linear equation in Λ^1_α with the solution [9]:

$$\Lambda^1_\alpha = -\frac{1}{4}\theta^{\mu\nu}\{A^0_\mu, \partial_\nu\alpha\} \,. \tag{3.4}$$

This solution is not unique, it is always possible to add solutions of the homogeneous equation $\Delta\Lambda^1 = 0$, we defer the discussion of such ambguities to section 5.

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We have introduced $\Delta\Lambda^k = i(\delta_\alpha \Lambda^k_\beta - \delta_\beta \Lambda^k_\alpha) + [\alpha, \Lambda^k_\beta] + [\Lambda^k_\alpha, \beta] - i\Lambda^k_{-i[\alpha,\beta]}$ as a shorthand (cp. [31]) in the inhomogeneous equation (3.3). The structure of $\Delta\Lambda^k$ is identical at every order k in $\hbar$.

The second order of (2.10) reads:

$$\Delta\Lambda^2 = \frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\mu\partial_\kappa\alpha, \partial_\nu\partial_\lambda\beta] - [\Lambda^1_\alpha, \Lambda^1_\beta] - \frac{i}{2}\theta^{\mu\nu}\left(\{\partial_\mu\Lambda^1_\alpha, \partial_\nu\beta\} - \{\partial_\mu\Lambda^1_\beta, \partial_\nu\alpha\}\right). \quad (3.5)$$

Using (3.4) for Λ^1_α, we find the following solution for (3.5):

$$\begin{aligned}\Lambda^2_\alpha = \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(&\{A^0_\mu, \{\partial_\nu A^0_\kappa, \partial_\lambda\alpha\}\} + \{A^0_\mu, \{A^0_\kappa, \partial_\nu\partial_\lambda\alpha\}\} + \\ &+ \{\{A^0_\mu, \partial_\nu A^0_\kappa\}, \partial_\lambda\alpha\}\} - \{\{F^0_{\mu\kappa}, A^0_\nu\}, \partial_\lambda\alpha\} - 2i[\partial_\mu A^0_\kappa, \partial_\nu\partial_\lambda\alpha]\Big). \quad (3.6)\end{aligned}$$

This solution differs from the one stated in [9] by a solution of the homogeneous solution at second order. Similar solutions had been found in [32] and [33].

Next we determine the field ψ (2.13) from (2.11) up to second order. To first order in $\hbar$ we obtain:

$$\Delta_\alpha\psi^1 := \delta_\alpha\psi^1 - i\alpha\psi^1 = i\Lambda^1_\alpha\psi^0 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi^0. \quad (3.7)$$

Note the definition of the operator $\Delta_\alpha\psi^k$ for any order k: $\Delta_\alpha\psi^k = \delta_\alpha\psi^k - i\alpha\psi^k$. Using (3.4) we find

$$\psi^1 = -\frac{1}{2}\theta^{\mu\nu}A^0_\mu\partial_\nu\psi^0 + \frac{i}{4}\theta^{\mu\nu}A^0_\mu A^0_\nu\psi^0. \quad (3.8)$$

Similarly the next order,

$$\Delta_\alpha\psi^2 = i\Lambda^2_\alpha\psi^0 + i\Lambda^1_\alpha\psi^1 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\Lambda^1_\alpha\partial_\nu\psi^0 - \frac{1}{2}\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu\psi^1 - \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu\partial_\kappa\alpha\partial_\nu\partial_\lambda\psi^0, \quad (3.9)$$

is solved by:

$$\begin{aligned}\psi^2 = &-\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\partial_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 + iA^0_\kappa A^0_\mu\partial_\nu\partial_\lambda\psi^0 - i\partial_\kappa A^0_\mu A^0_\nu\partial_\lambda\psi^0 + iF^0_{\kappa\mu}A^0_\nu\partial_\lambda\psi^0 - \\ &\qquad - iA^0_\nu\partial_\kappa A^0_\mu\partial_\lambda\psi^0 + 2iA^0_\nu F^0_{\kappa\mu}\partial_\lambda\psi^0 + 2A^0_\mu A^0_\kappa A^0_\nu\partial_\lambda\psi^0 - A^0_\mu A^0_\nu A^0_\kappa\partial_\lambda\psi^0\Big) - \\ &- \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(2\partial_\kappa A^0_\mu\partial_\lambda A^0_\nu\psi^0 - 2i\partial_\kappa A^0_\mu A^0_\lambda A^0_\nu\psi^0 + 2iA^0_\nu A^0_\lambda\partial_\kappa A^0_\mu\psi^0 + \\ &\qquad + i[[\partial_\kappa A^0_\mu, A^0_\nu], A^0_\lambda]\psi^0 + 4iA^0_\nu F^0_{\kappa\mu}A^0_\lambda\psi^0 - A^0_\kappa A^0_\lambda A^0_\mu A^0_\nu\psi^0 + \\ &\qquad + 2A^0_\kappa A^0_\mu A^0_\nu A^0_\lambda\psi^0\Big). \quad (3.10)\end{aligned}$$

Note that since we use a Λ^2_α different from the one stated in [9], (3.10) differs from the one stated there. The conjugate field $\overline{\psi} = \psi^\dagger\gamma^0$ is obtained by conjugation of ψ, $\overline{\psi}^0 = \overline{\psi^0}$ and $\overline{\psi}^1 = \overline{\psi^1}$.

The enveloping algebra gauge potential (2.14) is determined by expanding (2.11), using $\Delta_\alpha A^k_\sigma = \delta_\alpha A^k_\sigma - i[\alpha, A^k_\sigma]$:

$$\begin{aligned}\Delta_\alpha A^1_\sigma &= \partial_\sigma \Lambda^1_\alpha - i[A^0_\sigma, \Lambda^1_\alpha] + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^0_\sigma, \partial_\nu \alpha\}\,,\\ \Delta_\alpha A^2_\sigma &= \partial_\sigma \Lambda^2_\alpha - i[A^0_\sigma, \Lambda^2_\alpha] - i[A^1_\sigma, \Lambda^1_\alpha] + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^1_\sigma, \partial_\nu \alpha\} +\\ &\quad + \frac{1}{2}\theta^{\mu\nu}\{\partial_\mu A^0_\sigma, \partial_\nu \Lambda^1_\alpha\} + \frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}[\partial_\kappa\partial_\mu A^0_\sigma, \partial_\lambda\partial_\nu \alpha]\,. \end{aligned} \tag{3.11}$$

A solution for A^1_σ is:

$$A^1_\sigma = -\frac{1}{4}\theta^{\mu\nu}\left(\{A^0_\mu, \partial_\nu A^0_\sigma\} - \{F^0_{\mu\sigma}, A^0_\nu\}\right), \tag{3.12}$$

with $F^0_{\mu\nu}$ the Lie algebra field strength (2.5). Using (3.4) and (3.6), we obtain for A^2_σ:

$$\begin{aligned} A^2_\sigma = \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(&\{\{A^0_\kappa, \partial_\lambda A^0_\mu\}, \partial_\nu A^0_\sigma\} - \{\{F^0_{\kappa\mu}, A^0_\lambda\}, \partial_\nu A^0_\sigma\} - 2i[\partial_\kappa A^0_\mu, \partial_\lambda\partial_\nu A^0_\sigma] -\\ &- \{A^0_\mu, \{\partial_\nu F^0_{\kappa\sigma}, A^0_\lambda\}\} - \{A^0_\mu, \{F^0_{\kappa\sigma}, \partial_\nu A^0_\lambda\}\} + \{A^0_\mu, \{\partial_\nu A^0_\kappa, \partial_\lambda A^0_\sigma\}\} +\\ &+ \{A^0_\mu, \{A^0_\kappa, \partial_\nu\partial_\lambda A^0_\sigma\}\} + \{\{A^0_\kappa, \partial_\lambda F^0_{\mu\sigma}\}, A^0_\nu\} - \{\{\mathcal{D}^0_\kappa F^0_{\mu\sigma}, A^0_\lambda\}, A^0_\nu\} -\\ &- 2\{\{F^0_{\mu\kappa}, F^0_{\sigma\lambda}\}, A^0_\nu\} + 2i[\partial_\kappa F^0_{\mu\sigma}, \partial_\lambda A^0_\nu] - \{F^0_{\mu\sigma}, \{A^0_\kappa, \partial_\lambda A^0_\nu\}\} +\\ &+ \{F^0_{\mu\sigma}, \{F^0_{\kappa\nu}, A^0_\lambda\}\}\Big)\,. \end{aligned} \tag{3.13}$$

The enveloping algebra-valued covariant derivative $\mathcal{D}_\sigma\psi = \partial_\sigma\psi - iA_\sigma \star \psi$ is obtained immediately, e.g.

$$\mathcal{D}_\sigma\psi|_{\mathcal{O}(\hbar^1)} = \theta^{\mu\nu}\left(-\frac{1}{2}A^0_\mu\partial_\nu\mathcal{D}^0_\sigma\psi^0 + \frac{i}{4}A^0_\mu A^0_\nu\mathcal{D}^0_\sigma\psi^0 + \frac{1}{2}F^0_{\mu\sigma}\mathcal{D}^0_\nu\psi^0\right). \tag{3.14}$$

In order to express the enveloping algebra-valued field strength in terms of Lie algebra-valued quantities, we insert (3.12) and (3.13) into the definition (2.7). We obtain:

$$F^1_{\rho\sigma} = -\frac{1}{4}\theta^{\kappa\lambda}\left(\{A^0_\kappa, \partial_\lambda F^0_{\rho\sigma}\} - \{\mathcal{D}^0_\kappa F^0_{\rho\sigma}, A^0_\lambda\} - 2\{F^0_{\rho\kappa}, F^0_{\sigma\lambda}\}\right). \tag{3.15}$$

If we had determined $F^1_{\rho\sigma}$ from its covariant transformation behaviour $\delta_\alpha F_{\rho\sigma} = i[\Lambda_\alpha \stackrel{\star}{,} F_{\rho\sigma}]$, only the first two terms in (3.15) could have been reproduced.

In second order we obtain:

$$\begin{aligned} F^2_{\rho\sigma} = \frac{1}{32}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(&2\{\{A^0_\mu, \partial_\nu A^0_\kappa\}, \partial_\lambda F^0_{\rho\sigma}\} - 2\{\{F^0_{\mu\kappa}, A^0_\nu\}, \partial_\lambda F^0_{\rho\sigma}\} -\\ &- 2\{A^0_\kappa, \partial_\lambda\{\mathcal{D}^0_\mu F^0_{\rho\sigma}, A^0_\nu\}\} - 4\{A^0_\kappa, \partial_\lambda\{F^0_{\rho\mu}, F^0_{\sigma\nu}\}\} - 4i[\partial_\mu A^0_\kappa, \partial_\nu\partial_\lambda F^0_{\rho\sigma}] +\\ &+ i\{[\{A^0_\mu, \partial_\nu A^0_\kappa\}, F^0_{\rho\sigma}], A^0_\lambda\} - i\{[\{F^0_{\mu\kappa}, A^0_\nu\}, F^0_{\rho\sigma}], A^0_\lambda\} +\\ &+ i\{[A^0_\kappa, \{A^0_\mu, \partial_\nu F^0_{\rho\sigma}\}], A^0_\lambda\} - i\{[A^0_\kappa, \{\mathcal{D}^0_\mu F^0_{\rho\sigma}, A^0_\nu\}], A^0_\lambda\} -\\ &- 2i\{[A^0_\kappa, \{F^0_{\rho\mu}, F^0_{\sigma\nu}\}], A^0_\lambda\} + i\{[A^0_\kappa, F^0_{\rho\sigma}], \{A^0_\mu, \partial_\nu A^0_\lambda\}\} -\\ &- i\{[A^0_\kappa, F^0_{\rho\sigma}], \{F^0_{\mu\lambda}, A^0_\nu\}\} + 2\{\{\partial_\mu A^0_\kappa, \partial_\nu F^0_{\rho\sigma}\}, A^0_\lambda\} + \end{aligned}$$

$$
\begin{aligned}
&+ 2[\partial_\mu[A^0_\kappa, F^0_{\rho\sigma}], \partial_\nu A^0_\lambda] + 2\{\{\mathcal{D}^0_\mu F^0_{\rho\kappa}, A^0_\nu\}, F^0_{\sigma\lambda}\} - 2\{\{A^0_\mu, \partial_\nu F^0_{\rho\kappa}\}, F^0_{\sigma\lambda}\} + \\
&+ 4\{\{F^0_{\mu\rho}, F^0_{\nu\kappa},\}, F^0_{\sigma\lambda}\} + 2\{F^0_{\rho\kappa}, \{\mathcal{D}^0_\mu F^0_{\sigma\lambda}, A^0_\nu\}\} - 2\{F^0_{\rho\kappa}, \{A^0_\mu, \partial_\nu F^0_{\sigma\lambda}\}\} + \\
&+ 4\{F^0_{\rho\kappa}, \{F^0_{\mu\sigma}, F^0_{\nu\lambda}\}\} + 4i[\partial_\mu F^0_{\rho\kappa}, \partial_\nu F^0_{\sigma\lambda}] + \\
&+ 2\{A^0_\kappa, \partial_\lambda\{A^0_\mu, \partial_\nu F^0_{\rho\sigma}\}\}\Big) . \qquad (3.16)
\end{aligned}
$$

4. Constructing actions

We focus on constructing NC generalisations of the following lagrangians:[2]

$$
\begin{aligned}
&\text{Yang-Mills} && \mathcal{L}^0_{\mathrm{YM}} = F^0_{\rho\sigma} F^{0\rho\sigma} && (4.1)\\
&\text{Minimally coupled fermions} && \mathcal{L}^0_{\mathrm{MCF}} = i\overline{\psi^0}\gamma^\rho \mathcal{D}^0_\rho \psi^0 && (4.2)\\
&\text{Mass term for fermions} && \mathcal{L}^0_{\mathrm{MF}} = m\overline{\psi^0}\psi^0 . && (4.3)
\end{aligned}
$$

Although a fermionic mass term is not part of the Standard Model, it suffices to illustrate the properties of NC gauge theory. The remaining Standard Model lagrangians of a scalar mass term, a scalar quartic potential, minimally coupled kinetic terms for the scalar field and the Yukawa potential can be obtained almost immediately using the results of the above mentioned three cases. The only additional complication is the so-called hybrid Seiberg-Witten map (cp. [10]).

Requiring that the NC generalisations of (4.1) to (4.3) are the Standard Model lagrangians in zeroth order in $\hbar$, the most natural approach is to replace every field by its NC analogue. Then the NC lagrangians can be written using the $\star$-product and the expansions of the enveloping algebra-valued fields:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &= F_{\rho\sigma} \star F^{\rho\sigma} , && (4.4)\\
\mathcal{L}_{\mathrm{MCF}} &= i\overline{\psi} \star \gamma^\rho \mathcal{D}_\rho \psi , && (4.5)\\
\mathcal{L}_{\mathrm{MF}} &= m\overline{\psi} \star \psi . && (4.6)
\end{aligned}
$$

First we discuss mass terms and gauge coupling terms for fermion fields ψ transforming from the left. The zeroth orders coincide with the commutative lagrangians by definition, the first and second order read:

$$
m\overline{\psi} \star \psi|_{\mathcal{O}(\hbar^1)} = m\left(\overline{\psi^0} \star^1 \psi^0 + \overline{\psi^1}\psi^0 + \overline{\psi^0}\psi^1\right) = m\frac{i}{2}\theta^{\mu\nu}\mathcal{D}^0_\mu\overline{\psi}^0\mathcal{D}^0_\nu\psi^0 , \qquad (4.7)
$$

$$
\begin{aligned}
m\overline{\psi} \star \psi|_{\mathcal{O}(\hbar^2)} &= m\left(\overline{\psi^0} \star^2 \psi^0 + \overline{\psi^1} \star^1 \psi^0 + \overline{\psi^0} \star^1 \psi^1 + \overline{\psi^1}\psi^1 + \overline{\psi^2}\psi^0 + \overline{\psi^0}\psi^2\right)\\
&= m\left(-\frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\mu\mathcal{D}^0_\kappa\overline{\psi^0}\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0 - \frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\mu\overline{\psi^0}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0\right) . \qquad (4.8)
\end{aligned}
$$

In these expressions the Lie algebra-valued covariant derivative is evaluated on a conjugate field as $\mathcal{D}^0_\mu\overline{\psi^0} = \partial_\mu\overline{\psi^0} + i\overline{\psi^0}A^0_\mu$.

[2] We use the unusual convention that the trace over matrix indices for Yang-Mills is part of the integral, not of the lagrangian.

The minimal coupling of matter fields to gauge potentials reads:

$$
\begin{aligned}
i\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^1)} &= -\frac{1}{2}\theta^{\mu\nu}\mathcal{D}^0_\mu\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0 + \frac{i}{2}\theta^{\mu\nu}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}\mathcal{D}^0_\nu\psi^0\,, && (4.9)\\
i\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^2)} &= -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\mathcal{D}^0_\mu\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0 + \frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\overline{\psi^0}F^0_{\lambda\mu}\mathcal{D}^0_\nu\mathcal{D}^0_\rho\psi^0 - \\
&\quad -\frac{1}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\mathcal{D}^0_\kappa\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda\left(F^0_{\mu\rho}\mathcal{D}^0_\nu\psi^0\right) - \frac{1}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho\left(\mathcal{D}^0_\kappa F^0_{\mu\rho}\right)\mathcal{D}^0_\lambda\mathcal{D}^0_\nu\psi^0 - \\
&\quad -\frac{i}{8}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0 - \frac{i}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0\,. && (4.10)
\end{aligned}
$$

Next we consider the NC Yang-Mills lagrangian. The zeroth order again is by definition identical to its commutative counterpart. For higher orders we only need the $\theta^{\mu\nu}$-expansion at jth order $F^j_{\rho\sigma}$ with lower indices, for the dual field strength indices $F^{j\rho\sigma}$ can be raised with a formal metric (we assume a flat NC space):

$$
F_{\rho\sigma}\star F^{\rho\sigma} = g^{\alpha\rho}g^{\beta\sigma}F_{\alpha\beta}\star F_{\rho\sigma}\,, \qquad (4.11)
$$

therefore the order $F_{\rho\sigma}\star F^{\rho\sigma}$ vs. $F^{\rho\sigma}\star F_{\rho\sigma}$ is unimportant. In first order in $\hbar$ we obtain

$$
\begin{aligned}
F_{\rho\sigma}\star F^{\rho\sigma}|_{\mathcal{O}(\hbar^1)} &= \frac{i}{2}\theta^{\mu\nu}\mathcal{D}^0_\mu F^0_{\rho\sigma}\mathcal{D}^0_\nu F^{0\rho\sigma} + \frac{1}{2}\theta^{\mu\nu}\{\{F^0_{\rho\mu},F^0_{\sigma\nu}\},F^{0\rho\sigma}\} - \\
&\quad -\frac{1}{2}\theta^{\mu\nu}\{A^0_\mu,\partial_\nu(F^0_{\rho\sigma}F^{0\rho\sigma})\} + \frac{i}{4}\theta^{\mu\nu}\{A^0_\mu,[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\}. && (4.12)
\end{aligned}
$$

Similarly, we obtain in second order:

$$
\begin{aligned}
F_{\rho\sigma}\star F^{\rho\sigma}|_{\mathcal{O}(\hbar^2)} &= \theta^{\mu\nu}\theta^{\kappa\lambda}\times\\
&\times\Bigg(-\frac{1}{8}\mathcal{D}^0_\mu\mathcal{D}^0_\kappa F^0_{\rho\sigma}\mathcal{D}^0_\nu\mathcal{D}^0_\lambda F^{0\rho\sigma} + \frac{i}{4}[\mathcal{D}^0_\kappa\{F^0_{\rho\mu},F^0_{\sigma\nu}\},\mathcal{D}^0_\lambda F^{0\rho\sigma}] + \\
&\quad + \frac{i}{8}\{[\mathcal{D}^0_\kappa F^0_{\rho\mu},\mathcal{D}^0_\lambda F^0_{\sigma\nu}],F^{0\rho\sigma}\} + \frac{1}{4}\{\{\{F^0_{\mu\kappa},F^0_{\nu\rho}\},F^0_{\lambda\sigma}\},F^{0\rho\sigma}\} - \\
&\quad - \frac{1}{4}\{A^0_\kappa,\partial_\lambda\{\{F^0_{\rho\mu},F^0_{\sigma\nu}\},F^{0\rho\sigma}\}\} + \frac{i}{8}\{A^0_\kappa,[A^0_\lambda,\{\{F^0_{\rho\mu},F^0_{\sigma\nu}\},F^{0\rho\sigma}\}]\} - \\
&\quad - \frac{i}{4}\{A^0_\kappa,\partial_\lambda\left(\mathcal{D}^0_\mu F^0_{\rho\sigma}\mathcal{D}^0_\nu F^{0\rho\sigma}\right)\} - \frac{1}{8}\{A^0_\kappa,[A^0_\lambda,\left(\mathcal{D}^0_\mu F^0_{\rho\sigma}\mathcal{D}^0_\nu F^{0\rho\sigma}\right)]\} + \\
&\quad + \frac{1}{8}\{A^0_\kappa,\partial_\lambda\{A^0_\mu,\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}\} - \frac{i}{16}\{A^0_\kappa,[A^0_\lambda,\{A^0_\mu,\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}]\} - \\
&\quad - \frac{i}{16}\{A^0_\kappa,\partial_\lambda\{A^0_\mu,[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\}\} - \frac{1}{32}\{A^0_\kappa,[A^0_\lambda,\{A^0_\mu,[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\}]\} + \\
&\quad + \frac{1}{8}\{\{A^0_\kappa,\partial_\lambda A^0_\mu\},\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} - \frac{1}{16}\{\{A^0_\kappa,\partial_\mu A^0_\lambda\},\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} - \\
&\quad - \frac{i}{16}\{\{A^0_\kappa,\partial_\lambda A^0_\mu\},[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} + \frac{i}{32}\{\{A^0_\kappa,\partial_\mu A^0_\lambda\},[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} - \\
&\quad - \frac{i}{16}\{A^0_\kappa,[\{A^0_\mu,\partial_\nu A^0_\lambda\},\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} + \frac{i}{32}\{A^0_\kappa,[\{A^0_\mu,\partial_\lambda A^0_\nu\},\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} - \\
&\quad - \frac{1}{32}\{A^0_\kappa,[\{A^0_\mu,[A^0_\nu,A^0_\lambda]\},\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} - \frac{1}{32}\{\{A^0_\kappa,[A^0_\lambda,A^0_\mu]\},[A^0_\nu,\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)]\} - \\
&\quad - \frac{i}{16}\{\{A^0_\kappa,[A^0_\lambda,A^0_\mu]\},\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\} + \frac{i}{8}[\{\mathcal{D}^0_\kappa F^0_{\rho\sigma},F^0_{\mu\lambda}\},\mathcal{D}^0_\nu F^{0\rho\sigma}] - \\
&\quad - \frac{i}{8}[\partial_\kappa A^0_\mu,\partial_\lambda\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)] - \frac{1}{16}[\partial_\kappa A^0_\mu,\partial_\lambda[A^0_\nu,\left(F^0_{\rho\sigma},F^{0\rho\sigma}\right)]] - \\
&\quad - \frac{1}{16}\{A^0_\kappa,\{\partial_\mu A^0_\lambda,\partial_\nu\left(F^0_{\rho\sigma}F^{0\rho\sigma}\right)\}\} + \frac{1}{8}\{\{F^0_{\mu\rho},F^0_{\nu\sigma}\},\{F^{0\ \rho}_\kappa,F^{0\ \sigma}_\lambda\}\}\Bigg)\,. && (4.13)
\end{aligned}
$$

The construction of an action requires the definition of an integral. Integration is in general difficult to implement for NC spaces. It is misleading to expect that the integral is related to summing the field values over points, since NC spaces are "pointless". Therefore the usual intuitions about integration have to be dropped; specific desired properties have to be imposed. In particular, the integral should have the trace property, such that a gauge-covariant lagrangian implies a gauge invariant action. We demand:

$$\int \mathrm{d}x f \star g = \int \mathrm{d}x g \star f \,. \tag{4.14}$$

If the integral allows the use of Stokes' theorem, we may partially integrate the derivatives of the $\star$-product for $\theta^{\mu\nu} = const$ and obtain, because of antisymmetry and x-independence of $\theta^{\mu\nu}$,

$$\int \mathrm{d}x f \star g = \int \mathrm{d}x fg = \int \mathrm{d}x gf = \int \mathrm{d}x g \star f \,. \tag{4.15}$$

In addition to the integral, we need for the Yang-Mills action a trace over the matrix indices of the generators of the underlying non-abelian gauge group:

$$\mathcal{S}_{\mathrm{YM}} = \tilde{c}\,\mathrm{Tr} \int \mathrm{d}x F_{\rho\sigma} \star F^{\rho\sigma} \,. \tag{4.16}$$

Altogether this ensures gauge invariance of the Yang-Mills action:

$$\delta_\alpha \mathcal{S}_{\mathrm{YM}} = \tilde{c}\,\mathrm{Tr} \int \mathrm{d}x \left(i[\Lambda_\alpha \stackrel{\star}{,} F_{\rho\sigma}] \star F^{\rho\sigma} + i F_{\rho\sigma} \star [\Lambda_\alpha \stackrel{\star}{,} F^{\rho\sigma}] \right) = 0 \,, \tag{4.17}$$

Since the sum over generators is not performed, the numerical factor of $\mathcal{S}_{\mathrm{YM}}$ in (4.16) is an arbitrary real constant $\tilde{c}$. In the NC enveloping algebra gauge theory, the choice of trace for the NC Yang-Mills action is restricted only by $\frac{1}{g^2} = \sum_\rho c_\rho \mathrm{Tr}(\rho(T^a)\rho(T^a))$ (cp. [10, 34]). Here g is the coupling of the gauge group, ρ denotes a representation of the generators of the Lie algebra and the parameters c_ρ are restricted only by the mentioned condition.

Obviously (4.12) and (4.13) are not explicitly gauge covariant. This lack of covariance is cured in the action. Under an integral the non-covariant terms are rearranged into a field strength:

$$\mathrm{Tr} \int \mathrm{d}x \theta^{\kappa\lambda} \left(\{A^0_\kappa, \partial_\lambda X\} - \frac{i}{2} \{A^0_\kappa, [A^0_\lambda, X]\} \right) = \mathrm{Tr} \int \mathrm{d}x \theta^{\kappa\lambda} F^0_{\kappa\lambda} X \,. \tag{4.18}$$

Partial integration and the trace property lead to the following result for the $\hbar$-expanded Yang-Mills lagrangian:

$$\tilde{c}\,\mathrm{Tr} \int \mathrm{d}x F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^1)} = \tilde{c}\theta^{\mu\nu}\,\mathrm{Tr} \int \mathrm{d}x \left(2F^0_{\rho\mu} F^0_{\sigma\nu} F^{0\rho\sigma} - \frac{1}{2} F^0_{\mu\nu} F^0_{\rho\sigma} F^{0\rho\sigma} \right), \tag{4.19}$$

$$\tilde{c}\,\mathrm{Tr} \int \mathrm{d}x F_{\rho\sigma} \star F^{\rho\sigma}|_{\mathcal{O}(\hbar^2)} =$$

$$= \tilde{c}\theta^{\mu\nu}\theta^{\kappa\lambda}\,\mathrm{Tr} \int \mathrm{d}x \Big(\frac{1}{8} F^0_{\mu\nu} F^0_{\kappa\lambda} F^0_{\rho\sigma} F^{0\rho\sigma} - \frac{i}{4} F^0_{\mu\nu} (\mathcal{D}^0_\kappa F^0_{\rho\sigma})(\mathcal{D}^0_\lambda F^{0\rho\sigma}) -$$

$$
\begin{aligned}
&-\frac{1}{8}(\mathcal{D}^0_\mu\mathcal{D}^0_\kappa F^0_{\rho\sigma})(\mathcal{D}^0_\nu\mathcal{D}^0_\lambda F^{0\rho\sigma})+\frac{i}{2}(\mathcal{D}^0_\mu F^0_{\rho\kappa})(\mathcal{D}^0_\nu F^0_{\sigma\lambda})F^{0\rho\sigma}+\\
&+\frac{1}{2}F^0_{\mu\rho}F^0_{\nu\sigma}F^{0\ \rho}_\kappa F^{0\ \sigma}_\lambda+\frac{1}{2}F^0_{\mu\rho}F^0_{\nu\sigma}F^{0\ \sigma}_\kappa F^{0\ \rho}_\lambda-\\
&-\frac{1}{2}F^0_{\mu\nu}F^0_{\kappa\rho}F^0_{\lambda\sigma}F^{0\rho\sigma}-\frac{1}{2}F^0_{\kappa\rho}F^0_{\lambda\sigma}F^0_{\mu\nu}F^{0\rho\sigma}+\\
&+\frac{1}{2}(F^0_{\mu\kappa}F^0_{\nu\rho}F^0_{\lambda\sigma}+2F^0_{\nu\rho}F^0_{\mu\kappa}F^0_{\lambda\sigma}+F^0_{\lambda\sigma}F^0_{\nu\rho}F^0_{\mu\kappa})F^{0\rho\sigma}\Big)\,. \qquad (4.20)
\end{aligned}
$$

The fermionic mass term reads:

$$
m\int \mathrm{d}x\overline{\psi}\star\psi|_{\mathcal{O}(\hbar^1)}=-\frac{m}{4}\theta^{\mu\nu}\int \mathrm{d}x\overline{\psi^0}F^0_{\mu\nu}\psi^0\,, \qquad (4.21)
$$

$$
\begin{aligned}
m\int \mathrm{d}x\overline{\psi}\star\psi|_{\mathcal{O}(\hbar^2)}=m\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\bigg(&\frac{i}{8}\overline{\psi^0}(\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu\psi^0-\frac{1}{8}\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}\psi^0+\\
&+\frac{1}{32}\overline{\psi^0}F^0_{\kappa\lambda}F^0_{\mu\nu}\psi^0\bigg)\,, \qquad (4.22)
\end{aligned}
$$

and the minimally gauge coupled fermionic action is:

$$
i\int \mathrm{d}x\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^1)}=\theta^{\mu\nu}\int \mathrm{d}x\left(-\frac{i}{4}\overline{\psi}^0\gamma^\rho F^0_{\mu\nu}\mathcal{D}^0_\rho\psi^0-\frac{i}{2}\overline{\psi^0}\gamma^\rho F^0_{\rho\mu}\mathcal{D}^0_\nu\psi^0\right), \qquad (4.23)
$$

$$
\begin{aligned}
i\int \mathrm{d}x\overline{\psi}\star\gamma^\rho\mathcal{D}_\rho\psi|_{\mathcal{O}(\hbar^2)}=\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\bigg(&-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0-\frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0-\\
&-\frac{1}{8}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda\mathcal{D}^0_\rho\psi^0-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\kappa\mu}F^0_{\lambda\nu}\mathcal{D}^0_\rho\psi^0-\\
&-\frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0+\\
&+\frac{i}{32}\overline{\psi^0}\gamma^\rho F^0_{\kappa\lambda}F^0_{\mu\nu}\mathcal{D}^0_\rho\psi^0\bigg)\,. \qquad (4.24)
\end{aligned}
$$

5. Ambiguities of enveloping algebra gauge theory

We stated after equation (3.4) that there is an ambiguity in the construction of the enveloping algebra-valued gauge parameter Λ_α. In addition, there are ambiguities in constructing fields and the gauge potential. In this section we investigate these ambiguities in first and second order.

The ambiguity (3.4) has been discussed shortly in [9], along with a discussion of field redefinitions. Ambiguities in the Seiberg-Witten map have been discussed in [32, 35, 36] and [37]. However, these approaches have not discussed the meaning of the ambiguities for the construction of actions (the Yang-Mills action was discussed in [32]).

All ambiguities have to be solutions of homogeneous equations:

$$
\begin{aligned}
\Delta\Lambda^n&=0\,,\\
\Delta_\alpha\psi^n&=0\,,\\
\Delta_\alpha A^n_\sigma&=0\,. \qquad (5.1)
\end{aligned}
$$

There are essentially two types of ambiguities, which have been called covariant and gauge ambiguities, respectively. However, we do not want to cover the ambiguities encyclopedically in this note, rather discuss only those relevant for the actions.

We therefore demand the following requirements for the ambiguities relevant here. They should be

- *hermitean*, e.g. $\overline{\Lambda_\alpha} = \Lambda_\alpha$.
- *of the same index structure as the actions*: That means that we do not allow additional factors of a metric to lower the indices in $\theta^{\mu\nu}$, e.g. as in [38].
- *not derivative-valued*: Such terms, e.g. $\Lambda^1_\alpha = i\theta^{\mu\nu}\partial_\mu\alpha\partial_\nu$, are hermitean and can be used to consistently solve also higher-order terms (e.g. Λ^2_α). However, they should be discussed in a different context.

The only first-order hermitean ambiguity for Λ^1_α is therefore:

$$\Lambda^{1,c_1}_\alpha = ic_1\theta^{\mu\nu}[A^0_\mu, \partial_\nu\alpha], \qquad c_1 \in \mathbb{R}. \tag{5.2}$$

This ambiguity Λ^1_α leads to an additional term for the fermion field

$$\psi^{1,c_1} = -c_1\theta^{\mu\nu}A^0_\mu A^0_\nu\psi^0\,, \qquad \text{and} \qquad \overline{\psi^{1,c_1}} = c_1\theta^{\mu\nu}\overline{\psi}^0 A^0_\mu A^0_\nu\,, \tag{5.3}$$

and to an additional term for the gauge potential:

$$A^{1,c_1}_\rho = ic_1\theta^{\mu\nu}\left([\partial_\rho A^0_\mu, A^0_\nu] - i[[A^0_\rho, A^0_\mu], A^0_\nu]\right) = ic_1\theta^{\mu\nu}\left(\partial_\rho(A^0_\mu A^0_\nu) - i[A^0_\rho, A^0_\mu A^0_\nu]\right). \tag{5.4}$$

Adding up these terms, the additional fermionic lagrangians $\mathcal{L}^{1,c_1}_{\mathrm{MF}}$ and $\mathcal{L}^{1,c_1}_{\mathrm{MCF}}$ derived from (5.2) are identically zero. The field strength corresponding to (5.4) is

$$F^{1,c_1}_{\rho\sigma} = c_1\theta^{\mu\nu}[[F^0_{\rho\sigma}, A^0_\mu], A^0_\nu], \tag{5.5}$$

and the corresponding Yang-Mills action is therefore

$$\mathcal{S}^{1,c_1}_{\mathrm{YM}} = -c_1\tilde{c}\theta^{\mu\nu}\mathrm{Tr}\int \mathrm{d}x\; (A^0_\mu A^0_\nu F^0_{\rho\sigma}F^{0\rho\sigma} - F^0_{\rho\sigma}F^{0\rho\sigma}A^0_\mu A^0_\nu) = 0\,. \tag{5.6}$$

Therefore the ambiguity Λ^{1,c_1}_α does not contribute at all to the action.

Additional ambiguities at first order are solutions of the homogeneous equations (5.1) for ψ^1,

$$\psi^{1,c_2} = c_2\theta^{\mu\nu}F^0_{\mu\nu}\psi^0\,, \tag{5.7}$$

and the gauge potential A^1_ρ,

$$A^{1,c_3}_\rho = c_3\theta^{\mu\nu}\mathcal{D}^0_\rho F^0_{\mu\nu}\,. \tag{5.8}$$

That this is the only ambiguity w.r.t. the gauge potential can be seen using the Bianchi identity. The field strength corresponding to $A^{1c_3}_\rho$ is

$$F^{1,c_3}_{\rho\sigma} = i[\mathcal{D}^{1,c_3}_\rho, \mathcal{D}^0_\sigma] + i[\mathcal{D}^0_\rho, \mathcal{D}^{1,c_3}_\sigma] = -ic_3\theta^{\mu\nu}[F^0_{\rho\sigma}, F^0_{\mu\nu}]\,. \tag{5.9}$$

Since this field strength is a commutator, it does not contribute to the Yang-Mills action. There are no additional ambiguities than (5.9) for the field strength, since we demanded that it is strictly calculated from the gauge potential and not constructed as a solution of the transformation law. While the Yang-Mills action at first order is unambiguous, (5.7) and (5.8) introduce an ambiguity in the fermionic action:

$$\mathcal{S}^{1,c_2,c_3}_{\mathrm{MCF,\ MF}} = \int \mathrm{d}nx \left(2c_2 \theta^{\mu\nu} \overline{\psi^0} F^0_{\mu\nu} (i\gamma^\rho \mathcal{D}^0_\rho - m)\psi^0 + (c_2 + c_3)\theta^{\mu\nu} \overline{\psi^0} i\gamma^\rho (\mathcal{D}^0_\rho F^0_{\mu\nu})\psi^0 \right). \quad (5.10)$$

We add (5.10) to the fermionic action derived in the previous section (4.21) and (4.23):

$$\begin{aligned} \mathcal{S}^{1}_{\mathrm{MCF,\ MF}} + \mathcal{S}^{1,c_2,c_3}_{\mathrm{MCF,\ MF}} = \int \mathrm{d}^n x \Bigg(\Big(\underbrace{2c_2 - \frac{1}{4}}_{d_1} \Big) \theta^{\mu\nu} \overline{\psi^0} F^0_{\mu\nu} (i\gamma^\rho \mathcal{D}^0_\rho - m)\psi^0 - \\ - \frac{i}{2} \theta^{\mu\nu} \overline{\psi^0} \gamma^\rho F^0_{\rho\mu} \mathcal{D}^0_\nu \psi^0 + (\underbrace{ic_2 + c_3}_{d_2}) \theta^{\mu\nu} \overline{\psi^0} \gamma^\rho (\mathcal{D}^0_\rho F^0_{\mu\nu})\psi^0 \Bigg). \end{aligned}$$

Choosing $c_2 = 1/8$ and $c_3 = -i/8$ two terms can be set to zero, but choosing any other value for d_1 and d_2 will be just as consistent with the structure of the enveloping algebra gauge theory.

Since physics must not depend on a choice of parameters, any prediction based on a particular value of d_i is unphysical. This type of ambiguity w.r.t. the value of c_2 and c_3 or d_1 and d_2 is called a field redefinition [9, 37, 39]. We may conclude that the only physically relevant term in the NC fermionic action at first order in $\hbar$ is

$$\mathcal{S}^{1,\mathrm{relevant}}_{\mathrm{MCF,\ MF}} = -\frac{i}{2} \theta^{\mu\nu} \int \mathrm{d}^n x \overline{\psi^0} \gamma^\rho F^0_{\rho\mu} \mathcal{D}^0_\nu \psi^0 \,, \quad (5.11)$$

since the two other terms are proportional to a field redefinition. In particular, no mass term appears at first order in $\hbar$. To repeat, all terms of the NC Yang-Mills action at first order in $\theta^{\mu\nu}$ are physically relevant.

Next let us turn to the ambiguities appearing at second order. First we investigate the effect of first-order ambiguties at second order. The ambiguity in the gauge parameter Λ^{1,c_1}_α leads to the following additional terms in Λ^2_α

$$\begin{aligned} \Lambda^{2,c_1}_\alpha = {} & c_1 \theta^{\mu\nu} \theta^{\kappa\lambda} \Big(-\frac{i}{2} \{ [\partial_\mu A^0_\kappa, A^0_\lambda], \partial_\nu \alpha \} - \frac{1}{4} \{ [[A^0_\mu, \partial_\kappa \alpha], A^0_\nu], A^0_\lambda \} + \frac{1}{4} \{ [[A^0_\mu, A^0_\kappa], A^0_\nu], \partial_\lambda \alpha \} \Big) - \\ & - \frac{c_1^2 i}{4} \theta^{\mu\nu} \theta^{\kappa\lambda} [[A^0_\mu, A^0_\nu], [A^0_\kappa, \partial_\lambda \alpha]]. \end{aligned} \quad (5.12)$$

The gauge parameter ambiguities Λ^{1,c_1}_α and Λ^{2,c_1}_α lead to additional terms in ψ^{2,c_1}:

$$\begin{aligned} \psi^{2,c_1} = {} & c_1 \theta^{\mu\nu} \theta^{\kappa\lambda} \left(-\frac{i}{2} [\partial_\mu A^0_\kappa, A^0_\lambda] \partial_\nu \psi^0 + \frac{1}{2} A^0_\mu A^0_\nu A^0_\kappa \partial_\lambda \psi^0 - \frac{i}{4} A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda \psi^0 \right) + \\ & + \frac{c_1^2}{2} \theta^{\mu\nu} \theta^{\kappa\lambda} A^0_\mu A^0_\nu A^0_\kappa A^0_\lambda \psi^0 \,, \end{aligned} \quad (5.13)$$

and the gauge potential

$$
\begin{aligned}
A_\rho^{2,c_1} &= c_1\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(-\frac{i}{2}\{[\partial_\mu A_\kappa^0, A_\lambda^0]\partial_\nu A_\rho^0\} + \frac{1}{4}\{[[A_\mu^0, A_\kappa^0], A_\nu^0], (F_{\lambda\rho}^0 + \partial_\lambda A_\rho^0)\} - \\
&\qquad - \frac{1}{4}\{[[A_\mu^0, (F_{\kappa\rho}^0 + \partial_\kappa A_\rho^0)], A_\nu^0], A_\lambda^0\}\Big) - \\
&\quad -\frac{c_1^2}{4}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(i[[A_\mu^0, A_\nu^0], [\partial_\rho A_\kappa^0, A_\lambda^0]] + [[A_\mu^0, A_\nu^0], [[A_\rho^0, A_\kappa^0], A_\lambda^0]]\Big) . \qquad (5.14)
\end{aligned}
$$

Plugging these terms into the second order fermionic mass and minimally coupled action, all contributions (in c_1^2 and in c_1) drop out. Similarly these ambiguities in c_1 do not contribute to the Yang-Mills action.

We have analysed in addition three second-order ambiguities in Λ_α^2

$$
\begin{aligned}
\Lambda_\alpha^{2,c_4} &= c_4\theta^{\mu\nu}\theta^{\kappa\lambda}\left(\{\partial_\mu A_\kappa^0, \partial_\nu\partial_\lambda\alpha\} + i\{\partial_\mu A_\kappa^0, [\partial_\nu\alpha, A_\lambda^0]\}\right) , \\
\Lambda_\alpha^{2,c_5} &= c_5\theta^{\mu\nu}\theta^{\kappa\lambda}\{\partial_\mu A_\nu^0, [A_\kappa^0, \partial_\lambda\alpha]\} , \\
\Lambda_\alpha^{2,c_6} &= c_6\theta^{\mu\nu}\theta^{\kappa\lambda}\{[A_\mu^0, A_\nu^0], [A_\kappa^0, \partial_\lambda\alpha]\} . \qquad (5.15)
\end{aligned}
$$

These lead to the following terms for fields ψ:

$$
\begin{aligned}
\psi^{2,c_4} &= ic_4\theta^{\mu\nu}\theta^{\kappa\lambda}\partial_\mu A_\kappa^0\partial_\nu A_\lambda^0\psi^0 , \\
\psi^{2,c_5} &= c_5\theta^{\mu\nu}\theta^{\kappa\lambda}\left(i\{\partial_\mu A_\nu^0, A_\kappa^0 A_\lambda^0\}\psi^0 + 2A_\mu^0 A_\nu^0 A_\kappa^0 A_\lambda^0\psi^0\right) , \\
\psi^{2,c_6} &= c_6\theta^{\mu\nu}\theta^{\kappa\lambda}2A_\mu^0 A_\nu^0 A_\kappa^0 A_\lambda^0\psi^0 , \qquad (5.16)
\end{aligned}
$$

and for the gauge potential

$$
\begin{aligned}
A_\rho^{2,c_4} &= ic_4\theta^{\mu\nu}\theta^{\kappa\lambda}\left(\partial_\rho(\partial_\mu A_\kappa^0\partial_\nu A_\lambda^0) - i[A_\rho^0, (\partial_\mu A_\kappa^0\partial_\nu A_\lambda^0)]\right) , \\
A_\rho^{2,c_5} &= c_5\theta^{\mu\nu}\theta^{\kappa\lambda}\left(\partial_\rho\{\partial_\mu A_\nu^0, A_\kappa^0 A_\lambda^0\} - i[A_\rho^0, \{\partial_\mu A_\nu^0, A_\kappa^0 A_\lambda^0\}]\right) , \qquad (5.17) \\
A_\rho^{2,c_6} &= c_6\theta^{\mu\nu}\theta^{\kappa\lambda}2\left(\partial_\rho(A_\mu^0 A_\nu^0 A_\kappa^0 A_\lambda^0) - i[A_\rho^0, A_\mu^0 A_\nu^0 A_\kappa^0 A_\lambda^0]\right) . \qquad (5.18)
\end{aligned}
$$

One can show that all actions built from these fields are identically zero. Although the list of ambiguities at second order Λ_α^{2,c_4}, Λ_α^{2,c_5} and Λ_α^{2,c_6} is not exhaustive, we are lead to believe that such ambiguities in Λ_α are altogether irrelevant from the point of view of actions. We may concentrate on field redefinitions.

Next we investigate in which sense the first order ambiguities (field redefinitions) proportional to c_2 resp. c_3 reappear as additional ambiguities at second order. For example the ambiguity parametrised by c_2: $\psi^{1,c_2} = c_2\theta^{\mu\nu}F_{\mu\nu}^0\psi^0$ leads to the following c_2 parametrised lagrangians and actions in second order:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{MF}}^{2,c_2} &= \frac{ic_2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}(\mathcal{D}_\kappa^0(\overline{\psi^0}F_{\mu\nu}^0)\mathcal{D}_\lambda^0\psi^0 + \mathcal{D}_\kappa^0\overline{\psi^0}\mathcal{D}_\lambda^0(F_{\mu\nu}^0\psi^0)) + c_2^2\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}F_{\mu\nu}^0 F_{\kappa\lambda}^0\psi^0 , \\
\mathcal{S}_{\mathrm{MF}}^{2,c_2} &= \left(c_2^2 - \frac{c_2}{2}\right)\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\ \overline{\psi^0}F_{\mu\nu}^0 F_{\kappa\lambda}^0\psi^0 , \\
\mathcal{L}_{\mathrm{MCF}}^{2,c_2} &= -\frac{c_2}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\Big(\mathcal{D}_\kappa^0(\overline{\psi^0}F_{\mu\nu}^0)\gamma^\rho\mathcal{D}_\lambda^0\mathcal{D}_\rho^0\psi^0 + \mathcal{D}_\kappa^0\overline{\psi^0}\gamma^\rho\mathcal{D}_\lambda^0(\mathcal{D}_\rho^0(F_{\mu\nu}^0\psi^0)) -
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - i\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0 - i\overline{\psi^0}\gamma^\rho F^0_{\kappa\rho}\mathcal{D}^0_\lambda(F^0_{\mu\nu}\psi^0)\Big) + \\
&\quad + ic_2^2\theta^{\mu\nu}\theta^{\kappa\lambda}\overline{\psi^0}F^0_{\mu\nu}\gamma^\rho\mathcal{D}^0_\rho(F^0_{\kappa\lambda}\psi^0)\,, \\
\mathcal{S}^{2,c_2}_{\text{MCF}} &= i\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\bigg(\Big(c_2^2 - \frac{c_2}{4}\Big)\,\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}\mathcal{D}^0_\rho(F^0_{\kappa\lambda}\psi^0) - \frac{c_2}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\lambda}\mathcal{D}^0_\rho\psi^0 + \\
&\qquad + \frac{c_2}{2}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0 + \frac{c_2}{2}\overline{\psi^0}\gamma^\rho F^0_{\kappa\rho}\mathcal{D}^0_\lambda(F^0_{\mu\nu}\psi^0)\bigg)\,. \qquad (5.19)
\end{aligned}
$$

The gauge potential ambiguity proportional to c_3 leads in second order to

$$
A^{2,c_3}_\rho = c_3\frac{1}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\left(\{\partial_\mu(\mathcal{D}^0_\rho F^0_{\kappa\lambda}), A^0_\nu\} - \frac{i}{2}\{[A^0_\mu, (\mathcal{D}^0_\rho F^0_{\kappa\lambda})], A^0_\nu\}\right). \qquad (5.20)
$$

For the minimally coupled fermion action this means in second order:

$$
\begin{aligned}
\mathcal{L}^{2,c_3}_{\text{MCF}} &= \frac{ic_3}{2}\theta^{\mu\nu}\theta^{\kappa\lambda}\left(\mathcal{D}^0_\kappa\overline{\psi^0}\gamma^\rho\mathcal{D}^0_\lambda((\mathcal{D}^0_\rho F^0_{\mu\nu})\psi^0) + \overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\kappa\mathcal{D}^0_\rho F^0_{\mu\nu})\mathcal{D}^0_\lambda\psi^0\right), \qquad (5.21) \\
\mathcal{S}^{2,c_3}_{\text{MCF}} &= c_3\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\left(-\frac{1}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}(\mathcal{D}^0_\rho F^0_{\kappa\lambda})\psi^0 + \frac{i}{2}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\kappa\mathcal{D}^0_\rho F^0_{\mu\nu})\mathcal{D}^0_\lambda\psi^0\right).
\end{aligned}
$$

In addition we get a mixed term in c_2 and c_3

$$
\mathcal{S}^{2,c_2,c_3}_{\text{MCF}} = c_2c_3\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\overline{\psi^0}\gamma^\rho\{F^0_{\mu\nu}, (\mathcal{D}^0_\rho F^0_{\kappa\lambda})\}\psi^0. \qquad (5.22)
$$

Obviously, the field redefinitions at first order give additional action terms at second order. We try to understand which terms are physically relevant and compare with the fermionic actions at second order, (4.22) and (4.24):

$$
\begin{aligned}
&\int \mathrm{d}x\overline{\psi}\star(i\gamma^\rho\mathcal{D}_\rho - m)\psi|_{\mathcal{O}(\hbar^2)} = \\
&= \theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\bigg(\frac{1}{32}\overline{\psi^0}F^0_{\kappa\lambda}F^0_{\mu\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \frac{1}{8}\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + \\
&\qquad + \frac{i}{8}\overline{\psi^0}(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0 - \qquad (5.23) \\
&\qquad - \frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0 - \frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0 - \frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0\bigg).
\end{aligned}
$$

Since c_2 and c_3 are free parameters, we cannot use them to argue that some of the second-order action terms in (5.23) are physically irrelevant. But to this end we can investigate new second order field redefinitions:

$$
\begin{aligned}
\psi^{2,c_7} &= c_7\theta^{\mu\nu}\theta^{\kappa\lambda}F^0_{\mu\nu}F^0_{\kappa\lambda}\psi^0\,, \\
\psi^{2,c_8} &= c_8\theta^{\mu\nu}\theta^{\kappa\lambda}F^0_{\kappa\mu}F^0_{\lambda\nu}\psi^0\,, \\
\psi^{2,c_9} &= ic_9\theta^{\mu\nu}\theta^{\kappa\lambda}(\mathcal{D}^0_\mu F^0_{\nu\kappa})\mathcal{D}^0_\lambda\psi^0\,. \qquad (5.24)
\end{aligned}
$$

These lead to the following additional actions:

$$
\mathcal{S}^{2,c_7}_{\text{MCF, MF}} = c_7\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\left(2\overline{\psi^0}F^0_{\mu\nu}F^0_{\kappa\lambda}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + i\overline{\psi^0}\gamma^\rho\{\mathcal{D}^0_\rho F^0_{\mu\nu}, F^0_{\kappa\lambda}\}\psi^0\right),
$$

$$\mathcal{S}^{2,c_8}_{\text{MCF, MF}} = c_8\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\,\Big(2i\overline{\psi^0}F^0_{\kappa\mu}F^0_{\lambda\nu}(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 + i\overline{\psi^0}\gamma^\rho\{\mathcal{D}^0_\rho F^0_{\kappa\mu}, F^0_{\lambda\nu}\}\psi^0\Big) \tag{5.25}$$

$$\mathcal{S}^{2,c_9}_{\text{MCF, MF}} = c_9\theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\,\Big(2i\overline{\psi^0}(\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu(i\gamma^\rho\mathcal{D}^0_\rho - m)\psi^0 - \overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\rho\mathcal{D}^0_\kappa F^0_{\lambda\mu})\mathcal{D}^0_\nu\psi^0\Big)\,.$$

The ambiguities (5.24) show that the first three terms in (5.23), especially all fermionic mass terms, are of the type of a field redefinition and therefore should vanish. The additional terms in the action introduced in (5.25) can be cancelled by additional redefinitions of the gauge potential, e.g. $A^{2,c_{10}}_\rho = c_{10}\theta^{\mu\nu}\theta^{\kappa\lambda}\{\mathcal{D}^0_\rho F^0_{\mu\nu}, F^0_{\kappa\lambda}\}$ etc. In contrast, the last four terms in (5.23) are not affected, they are not of the type of field redefinitions. Therefore the only physically relevant terms in the fermionic action to second order are

$$\begin{aligned}
&\int \mathrm{d}x\overline{\psi}\star(i\gamma^\rho\mathcal{D}_\rho - m)\psi|_{\mathcal{O}(\hbar^2)} = \\
&= \theta^{\mu\nu}\theta^{\kappa\lambda}\int \mathrm{d}x\Big(-\frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\kappa}F^0_{\lambda\rho}\mathcal{D}^0_\nu\psi^0 - \frac{i}{4}\overline{\psi^0}\gamma^\rho F^0_{\mu\rho}F^0_{\nu\kappa}\mathcal{D}^0_\lambda\psi^0 - \\
&\qquad - \frac{1}{4}\overline{\psi^0}\gamma^\rho(\mathcal{D}^0_\mu F^0_{\kappa\rho})\mathcal{D}^0_\nu\mathcal{D}^0_\lambda\psi^0 - \frac{i}{8}\overline{\psi^0}\gamma^\rho F^0_{\mu\nu}F^0_{\kappa\rho}\mathcal{D}^0_\lambda\psi^0\Big)\,.
\end{aligned}$$

The Yang-Mills action is not affected by these field redefitions,[3] since the field stength corresponding e.g. to $A^{2,c_{10}}_\rho$, again only results in a commutator term in the action.

6. Conclusion

From the point of view of the theory of consistent deformations (cp. [40]), the enveloping algebra is a nontrivial deformation of type 1. This means that although the gauge transformation is trivially deformed ($\delta_{\Lambda_\alpha} \equiv \delta_\alpha$), the deformation is non-trivial, it cannot only be obtained via field redefinitions. Indeed we have seen that both the Yang-Mills action and the fermionic interaction term (5.11) are nontrivial at first and second order.

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